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INFINITE INFLATIONS OF CRUMPLED CUBES

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Based upon recent results characterizing Q-manifolds, this paper sets forth an explicit method for retooling certain pathology arising in finite dimensional manifolds as comparable pathology in the Hilbert cube Q. In particular, with reference to an example constructed by the author and J.J. Walsh, it presents an upper semicontinuous decomposition G of Q into points and a null sequence of cellular arcs such that the associated decomposition space is not a Q-manifold, and it also provides a new procedure for embedding finite dimensional compacta as wild subsets of Q.

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Cellular decomposition	Q-manifold	Disjoint k-cells Property
infinite codimension	infinite inflation	absolute neighborhood retract
crumpled n-cube	wild embedding	nonshrinkable decomposition

This paper, which displays some elementary embedding and decomposition phenomena in Q, was inspired by finite dimensional examples and methods. In effect, it transforms a certain finite dimensional technique into a similar but new infinite dimensional one, and then it applies that technique to transform, rather effortlessly, certain finite dimensional pathological examples into similar infinite dimensional ones.

The technique explored and exploited here is that of inflating a crumpled cube, which was first introduced in [10]. Given a crumpled n-cube C in Euclidean n-space E_n (C is the closure of the bounded complementary domain determined by an $(n-1)$-sphere S topologically embedded in E^n) such that Int C fails to be simply connected, one often can inflate C to a crumpled $(n+1)$-cube C^* in E^{n+1} such that Int C^* also fails to be simply connected and, in addition, the set of points where the n-sphere S^* bounding C^* is wildly embedded coincides topologically with the set of points where S is wildly embedded. In this sense, inflation does not compound wildness.

This technique is revised somewhat in what follows to permit the inflating of a crumpled n-cube C to an infinite dimensional object, which, under mild conditions on C, is equivalent to Q. As the primary benefit of this operation, the boundary Bd C of C (Bd C denotes the $(n-1)$-sphere in E^n bounding C), which embeds naturally in

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the infinite dimensional object, will not be nicely embedded there (that is, cannot be embedded as a Z-set) as long as Bd C is not nicely embedded in C. It is a direct consequence that certain wild embeddings of finite dimensional spaces in E^n, namely, some embeddings with image in the boundary of a crumpled n-cube, readily lead to wild embeddings of the same spaces in Q. It is also a direct consequence that a finite dimensional example constructed by Daverman and Walsh [12] readily leads to an example of a decomposition K^* of Q into points and a null sequence of cellular arcs such that Q/K^* fails to be a Q-manifold.

At the heart of the arguments here stand some recently established characterizations of Q-manifolds. Using R.D. Edwards' result (cf. [9, Ch. XIV]) that the product of Q with any locally compact (metric) ANR is a Q-manifold, H. Toruńczyk [18] proved that a locally compact ANR is a Q-manifold if and only if it satisfies the Disjoint k-cells Property for all integers $k \geq 0$. (One says that a metric space X satisfies the Disjoint k-cells Property if any two maps of the k-cell I^k can be approximated, in the sup-norm metric, by maps having disjoint images.) Daverman and Walsh [11], in turn, used Toruńczyk's result to prove that a locally compact ANR is a Q-manifold if and only if (i) it satisfies the Disjoint 2-cells Property and (ii) roughly, it has many finite dimensional subsets. An explicit consequence of their main therorem needed for this work is the following [11, Theorem 7.1].

Theorem X. *Let $\pi : M \to X$ be a cell-like map of a compact Q-manifold M onto an ANR X such that*

(i) *X has the Disjoint 2-cells Property, and*

(ii) *$\pi(N_\pi)$ has infinite codimension in X.*

Then X is a Q-manifold.

To explain further the hypotheses of Theorem X, N_π denotes the union of all the nondegenerate sets $\pi^{-1}(x)$, $x \in X$; a closed subset N of X has *infinite codimension* in X if, for all integers $q \geq 0$ and all open sets U of X, $H_q(U, U-N)=0$, and an arbitrary subset F of X has infinite codimension in X if every closed subset N of X contained in F does.

1. Description of infinite inflations

Throughout this section let C denote a crumpled n-cube in E^n. Of course, C might be an n-cell, but more interesting phenomena arise when it is not. In [10] the *inflation* Infl(C) of C is defined as

$$\{(c, t) \in E^n \times E^1 = E^{n+1} \mid c \in C, |t| \leq f(c)\}$$

where $f : C \to [0, \infty)$ is a continuous function such that Bd $C = f^{-1}(0)$. Frequently Infl C turns out to be a crumpled $(n+1)$-cube; the only issue concerns whether its frontier, topologically the space obtained from two copies of C after identifying corresponding points from their boundaries, is an n-sphere. In case it is, the frontier of Infl C (=Bd Infl C) can reveal some unusual embedding properties of n-spheres in E^{n+1}.

Abstractly, however, Infl C coincides with $(C\times I)/G$, where G denotes the upper semicontinuous decomposition of $C\times I$ into points and the arcs $c\times I$, $c\in \mathrm{Bd}\, C$. In view of this, we shall speak of the *infinite inflation* $J^\infty(C)$ *of* C as

$$J^\infty(C)=(C\times Q)/G,$$

where G denotes the decomposition of $C\times Q$ into points and the sets $c\times Q$, $c\in \mathrm{Bd}\, C$. Furthermore, we shall consider Bd C to be naturally embedded in $J^\infty(C)$ under

$$\mathrm{Bd}\, C\to \mathrm{Bd}\, C\times \mathrm{pt}\subset C\times Q\to J^\infty(C).$$

Without any restrictions on C, $J^\infty(C)$ is always an AR (absolute retract).

Proposition 1. *For each crumpled n-cube C, $J^\infty(C)$ is an AR.*

Proof. Instead of regarding $J^\infty(C)$ as a decomposition space, view it as the adjunction space Bd $C\cup_p (C\times Q)$ obtained from the disjoint union of Bd C and $C\times Q$ by matching (Bd C)$\times Q$ with Bd C via the projection map $p:\mathrm{Bd}\, C\times Q\to \mathrm{Bd}\, C$. Since all the sets and subsets named are ANR's, a classical theorem of Borsuk [7, Theorem 9.1, p. 116] implies that $J^\infty(C)$ is also an ANR.

According to a result of Bing [6], C itself is contractible. To prove that $J^\infty(C)$ is an AR, all that remains to be done is to show that it is contractible, and there are many ways to do that. For example, $J^\infty(C)$ is the image of the contractible $C\times Q$ under a cell-like map π, and, because both $J^\infty(C)$ and $C\times Q$ are ANR's, π must be a homotopy equivalence [14, 15]. Alternatively, $C\times Q$ deformation retracts to $C\times$ point by a deformation that preserves C-coordinates, thereby inducing a deformation of $J^\infty(C)$ to a copy of C and implying directly that $J^\infty(C)$ is contractible.

2. Characterization of the inflations that yield Q

Fundamental to this paper is the observation that, even if C fails to be an n-cell, $J^\infty(C)$ can be homeomorphic to Q. The next result provides a special Disjoint 2-cells Property in C characterizing when this occurs.

Theorem 2. *Let C denote a crumpled n-cube. Then $J^\infty(C)$ is homeomorphic to Q if and only if any two maps $f_1, f_2: B^2\to C$ can be approximated, arbitrarily closely, by maps f_1', f_2' such that*

$$f_1'(B^2)\cap f_2'(B^2)\cap \mathrm{Bd}\, C=\emptyset.$$

Proof. By way of notation, let $e : C \to J^\infty(C)$ denote a natural embedding, as before, sending C to the image in $J^\infty(C)$ of $C \times \text{point} \subset C \times Q$, and let $R : J^\infty(C) \to C$ denote the map induced by the projection $p : C \times Q \to C$ so as to make the diagram below commute.

$$\begin{tikzcd} C \times Q \arrow[d, "\pi"] \arrow[dr, "p"] & \\ J^\infty(C) \arrow[r, "R"] & C \end{tikzcd}$$

It is important to note that Re is the identity on C and that $R^{-1}(c)$ is a singleton whenever $c \in \mathrm{Bd}\, C$, in other words, that R is 1–1 over $\mathrm{Bd}\, C \subset J^\infty(C)$.

First, for the elementary half of the argument, suppose $J^\infty(C)$ to be homeomorphic to Q. Given two maps $f_1, f_2 : B^2 \to C$, then one can certainly approximate the maps $ef_1, ef_2 : B^2 \to J^\infty(C)$ by disjoint maps (embeddings) $f_1^*, f_2^* : B^2 \to J^\infty(C)$ so closely that Rf_1^* and Rf_2^* are close to f_1 and f_2, respectively. Since R is 1–1 over $\mathrm{Bd}\, C$, the disjointness of $f_1^*(B^2)$ and $f_2^*(B^2)$ guarantees that

$$Rf_1^*(B^2) \cap Rf_2^*(B^2) \cap \mathrm{Bd}\, C = \emptyset.$$

Next, for the reverse implication, we shall suppose C has this special Disjoint 2-cells Property and shall make use of Theorem X. The crucial step is an argument proving that $J^\infty(C)$ has the Disjoint 2-cells Property. To that end, consider two maps $\mu_1, \mu_2 : B^2 \to J^\infty(C)$ and $\varepsilon > 0$. Since $\pi : C \times Q \to J^\infty(C)$ is a cell-like map between AR's, it follows from lifting properties of such maps [3, 16] that there exist maps $F_1, F_2 : B^2 \to C \times Q$ such that $\mathrm{dist}\,(\pi F_e, \mu_e) < \frac{1}{2}\varepsilon$ for $e = 1, 2$. Let f_e denote F_e followed by the projection of $C \times Q$ to C, and let k_e denote F_e followed by the projection of $C \times Q$ to Q. By hypothesis, there exist maps $f'_e : B^2 \to C$ approximating f_e $(e = 1, 2)$ such that

$$f'_1(B^2) \cap f'_2(B^2) \cap \mathrm{Bd}\, C = \emptyset,$$

and by properties of Q there exist maps $k'_e : B^2 \to Q$ approximating k_e $(e = 1, 2)$ such that $k'_1(B^2) \cap k'_2(B^2) = \phi$. More precisely, both f'_e and k'_e can be chosen so close to f_e and k_e that the maps F'_e defined as $f'_e \times k'_e$ satisfy

$$\mathrm{dist}\,(\pi F'_e, \pi F_e) < \tfrac{1}{2}\varepsilon \quad (e = 1, 2).$$

Clearly, the maps $\pi F'_e$ are approximations to μ_e $(e = 1, 2)$ having disjoint images, as required.

The other step in the argument involves showing that $\pi(N_\pi)$, where N_π denotes the union of the nondegenerate sets $\pi^{-1}(z)$, $z \in J^\infty(C)$, has infinite codimension in $J^\infty(C)$. In this situation, $\pi(N_\pi) = \mathrm{Bd}\, C \subset J^\infty(C)$. Moreover, $\mathrm{Bd}\, C$ has infinite codimension in C, essentially because $\mathrm{Bd}\, C$ serves as the boundary of the generalized n-manifold-with boundary C (cf. [2, Theorem IV, 4]). As in the proof of [11, Lemma 2.2], Bd $C \times Q$ then has infinite codimension in $C \times Q$. Since π is a cell-like map, the Vietoris–Begle Mapping Theorem [4] shows that, for any open subset U of $J^\infty(C)$,

$$\pi_* : H_*(\pi^{-1}(U), \pi^{-1}(U) - (\mathrm{Bd}\, C \times Q)) \to H_*(U, U - \mathrm{Bd}\, C)$$

is an isomorphism, implying that Bd $C = \pi(N_\pi)$ has infinite codimension in $J^\infty(C)$.

As a result, Theorem X establishes that $J^\infty(C)$ is homeomorphic to Q.

Corollary 3. *Let C denote a crumpled n-cube ($n \geq 5$). Then $J^\infty(C)$ is homeomorphic to Q if and only if C has the Disjoint 2-cells Property.*

Proof. This will result from Theorem 2, after we show that the special Disjoint 2-cells Property given there is equivalent to the usual one. For the less trivial portion of this equivalence, consider two maps of B^2 to C and apply the special Disjoint 2-cells Property to obtain approximations $f_1, f_2 : B^2 \to C$ for which $f_1(B^2) \cap f_2(B^2) \cap \mathrm{Bd}\, C = \emptyset$. But then these images can intersect only at points of Int C, and the fact that Int C is an n-manifold, $n \geq 5$, implies that f_1 and f_2 can be further approximated by "general position" maps whose images are entirely disjoint.

Combining Theorem 2 with results of [13] in case $n = 3$ or with those of [10] in case $n \geq 5$ leads to the following corollary.

Corollary 4. *Let C denote a crumpled n-cube ($n \neq 4$). Then $J^\infty(C)$ is homeomorphic to Q if and only if S^n is homeomorphic to $C \cup_{\mathrm{Id}} C$, which is the space resulting from the disjoint union of two copies of C after identifying the corresponding points in their boundaries.*

3. First application: wild embeddings in Q

Wild embeddings of reasonable spaces are usually diagnosed by the failure of local simple connectedness in their complements. Given a subspace A of a space X and a point $a \in \mathrm{Cl}(X - A)$, one writes that $X - A$ is a 1-LC at a (meaning, is locally 1-connected at a) if for each neighborhood U of a there exists a neighborhood V of a such that every map $\mathrm{Bd}\, I^2 \to V \cap (X - A)$ can be extended to a map $I^2 \to U \cap (X - A)$.

Proposition 5. *Let C be a crumpled n-cube and let X be a closed subset of* Bd C *such that $C - X$ fails to be 1-LC at some point of X ($C - X$ fails to be simply connected). Then, under the natural embedding $e : X \to \mathrm{Bd}\, C \to J^\infty(C)$, $J^\infty(C) - e(X)$ fails to be 1-LC at some point of $e(X)$ ($J^\infty(C) - e(X)$ fails to be simply connected).*

Proof. The embedding e extends to an embedding $e : C \to J^\infty(C)$ such that, for the map $R : J^\infty(C) \to C$ defined at the outset of Theorem 2, Re is the identity on C.

The next result is not new, but the methodology is. R.Y.T. Wong gave the first example of a wildly embedded Cantor set in Q [19].

Corollary 6. *There exists an embedding e of the Cantor set X in Q such that $Q-e(X)$ fails to be simply connected.*

Proof. The crumpled 3-cube C bounded by the Alexander Horned Sphere [1] has a Cantor set X of bad points in Bd C such that $C-X$ fails to be simply connected. By Proposition 5, $J^{\infty}(C)-e(X)$ also fails to be simply connected. Moreover, by Theorem 2, $J^{\infty}(C)$ is homeomorphic to Q, for it follows quite directly, as in Theorem 9 of [13], that C has the special Disjoint 2-cells Property; on the other hand, the same result also can be derived from Corollary 4 and the theorem of Bing [5] that $C \cup_{\mathrm{Id}} C = S^3$.

In similar fashion, one can obtain other corollaries giving wild embeddings of other finite dimensional compacta X in Q so as to have nonsimply connected complement.

4. Second application: non-shrinkable decompositions of Q

Throughout this section C will denote a crumpled n-cube, K will denote a cell-like upper semicontinuous decomposition of C such that every nondegenerate element lies in Bd C, and π will denote the decomposition map $C \to C/K$.

In this setting K induces a decomposition $J^{\infty}(K)$ of $J^{\infty}(C)$, namely, the one into the sets $k \subset \mathrm{Bd}\, C \subset J^{\infty}(C)$, $k \in K$, and the other singletons $\{x\} \subset J^{\infty}(C) - \mathrm{Bd}\, C$. Clearly, $J^{\infty}(K)$ is also cell-like. It is particularly convenient to also regard $J^{\infty}(C)/J^{\infty}(K)$ as an infinite inflation $J^{\infty}(C/K)$ of C/K, with

$$J^{\infty}(C)/J^{\infty}(K) = J^{\infty}(C/K) = ((C/K)\times Q)/G^*,$$

where G^* denotes the decomposition of $(C/K)\times Q$ into points and the sets $\pi(c)\times Q$, $c \in \mathrm{Bd}\, C$.

Proposition 7. *There exists an embedding e^* of C/K in $J^{\infty}(C/K)$ and there exists a map R^* of $J^{\infty}(C/K)$ to C/K such that R^*e^* is the identity on C/K and R^* is 1–1 over $\pi(Bd\, C) \subset C/K$.*

Proposition 7 follows by a construction comparable to that given at the start of the proof to Theorem 2.

Corollary 8. *The decomposition space $J^{\infty}(C/K) = J^{\infty}(C)/J^{\infty}(K)$ is an AR if and only if C/K is an AR.*

Proof. Proposition 7 guarantees that C/K is homeomorphic to a retract of

$J^\infty(C/K)$. As a result, C/K is an AR if $J^\infty(C/K)$ is. On the other hand, if C/K is an AR, then the fact that $J^\infty(C/K)$ is also an AR follows either by the argument given in Proposition 1 or by the combination of Proposition 1 and [7, 9.17, p. 121].

Theorem 9. *The inflation $J^\infty(C/K)$ is homeomorphic to Q if and only if C/K is an AR and any two maps $f_1, f_2 : B^2 \to C/K$ can be approximated, arbitrarily closely, by maps $f_1', f_2' : B^2 \to C/K$ such that*

$$f_1'(B^2) \cap f_2'(B^2) \cap \pi(\mathrm{Bd}\, C) = \emptyset.$$

The argument proceeds almost identically to the latter parts of the one traced out in Theorem 2.

A closed subset A of a Q-manifold M is said to be *cellular in M* provided A has arbitrarily small closed neighborhoods N such that both N and $\mathrm{Fr}\, N$ are homeomorphic to Q and $\mathrm{Fr}\, N$ is a Z-set in N. If A is known to be cell-like, it will be cellular if and only if M/A is homeomorphic to M (cf. [8]). In addition, according to Čerin [8], the cellularity of cell-like sets in Q is detected there by the literal analogue of McMillan's Cellularity Criterion [17]. In the situation at hand, the next theorem guarantees that the cellularity in Q of a cell-like set $A \subset \mathrm{Bd}\, C \subset J^\infty(C) = Q$ is detected as well by a property of maps into C which could be translated into the requirement that A satisfy McMillan's Cellularity Criterion in C.

Proposition 10. *Suppose C is a crumpled n-cube ($n \geq 3$) such that $J^\infty(C)$ is homeomorphic to Q and K is a cell-like upper semicontinuous decomposition of C such that each nondegenerate $k \in K$ lies in* Bd C. *Then $J^\infty(K)$ is a cellular decomposition of $J^\infty(C)$ if and only if, for each $k \in K$, each map $f : B^2 \to C/k$ can be approximated by a map $f' : B^2 \to C/k$ whose image misses the point corresponding to k.*

Proof. The forward implication being relatively routine, we consider only the reverse. The case of interest concerns a nondegenerate $k \in J^\infty(K)$, which corresponds to some $k \in K$ contained in Bd C.

To show that k is cellular in $Q = J^\infty(C)$, it suffices to show that Q/k is homeomorphic to Q. Since

$$Q/k = J^\infty(C)/J^\infty(k) = J^\infty(C/k),$$

and since C/k is an AR by results of Kozlowski [15], it will suffice to prove that C/k has the special Disjoint 2-cells Property of Theorem 9. Any two maps $f_1, f_2 : B^2 \to C/k$ can be approximated first by maps $f_1', f_2' : B^2 \to C/k$ whose images miss the point corresponding to k. Thus, f_1', f_2' can be regarded as maps into $C - k$. Based on the hypothesis $J^\infty(C) = Q$, Theorem 2 imples that f_1', f_2' can be further approximated by maps $f_1'', f_2'' : B^2 \to C/k$ with the described property.

The preceding work in this section is designed to apply to the following example of Daverman–Walsh.

Example [12]. There exists an upper semicontinuous decomposition K of a crumpled 3-cube C into points and a null sequence (that is, for each $\varepsilon > 0$, only a finite number of the sets have diameter $\geqslant \varepsilon$) of cellular arcs in Bd C such that

(1) $C \cup_{\mathrm{Id}} C$ is topologically S^3,

(2) C/K does not embed in any 3-manifold (in particular, C/K does not satisfy the mapping property of Theorem 9), and

(3) for each $k \in K$, C/k satisfies the mapping property given in Proposition 10.

Corollary 11. *There exists an upper semicontinuous decomposition K^* of Q into points and a null sequence of cellular arcs such that Q/K^* fails to be a Q-manifold.*

Proof. By Corollary 4, the inflation $J^\infty(C)$ of the crumpled cube C from the Example is homeomorphic to Q. The promised decomposition K^*, of course, is $J^\infty(K)$. By Proposition 10, the arcs of K^* are cellular. By Theorem 9, $J^\infty(C)/J^\infty(K) = J^\infty(C/K)$ is not homeomorphic to Q. Moreover, $J^\infty(C/K)$ cannot be a Q-manifold, for otherwise Chapman's Triangulation and Classification Theorems [9, Theorems 36.2 & 38.1] would show it to be Q itself. (Those who prefer a simpler direct argument might review the proof of Proposition 2 to see why, if $J^\infty(C/K)$ were a Q-manifold, C/K would necessarily satisfy the mapping property of Theorem 9.)

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